Exercise 34

A thin wire has the shape of the first-quadrant part of the circle with center the origin and radius a. If the density function is $\rho(x, y) = kxy$, find the mass and center of mass of the wire.

Solution

Begin by parameterizing the wire's position as a function of t: $x(t) = a\cos t$ and $y(t) = a\sin t$ with $0 \leq t \leq \pi/2$. Integrate the density over the wire's length in order to get the wire's mass.

$$\begin{aligned}
m &= \int_C \rho\, ds \\
&= \int_0^{\pi/2} kx(t)y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= k\int_0^{\pi/2} (a\cos t)(a\sin t)\sqrt{(-a\sin t)^2 + (a\cos t)^2}\, dt \\
&= ka^2 \int_0^{\pi/2} \frac{1}{2}\sin 2t\sqrt{a^2\sin^2 t + a^2\cos^2 t}\, dt \\
&= \frac{ka^2}{2}\int_0^{\pi/2} \sin 2t\sqrt{a^2(\sin^2 t + \cos^2 t)}\, dt \\
&= \frac{ka^3}{2}\int_0^{\pi/2} \sin 2t\, dt \\
&= \frac{ka^3}{2}(1) \\
&= \frac{ka^3}{2}.
\end{aligned}$$

Calculate the x-coordinate of the center of mass.

$$\bar{x} = \frac{\int x\,dm}{\int dm} = \frac{\int_C x(\rho\,ds)}{\int_C \rho\,ds} = \frac{\int_0^{\pi/2} x(t)[kx(t)y(t)]\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt}{\int_0^{\pi/2} kx(t)y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt}$$

$$= \frac{k\int_0^{\pi/2}(a\cos t)^2(a\sin t)\sqrt{(-a\sin t)^2 + (a\cos t)^2}\,dt}{k\int_0^{\pi/2}(a\cos t)(a\sin t)\sqrt{(-a\sin t)^2 + (a\cos t)^2}\,dt}$$

$$= \frac{\int_0^{\pi/2}(a^3\cos^2 t\sin t)\sqrt{a^2\sin^2 t + a^2\cos^2 t}\,dt}{\int_0^{\pi/2}(a^2\cos t\sin t)\sqrt{a^2\sin^2 t + a^2\cos^2 t}\,dt}$$

$$= \frac{\int_0^{\pi/2}(a^4\cos^2 t\sin t)\,dt}{\int_0^{\pi/2}(a^3\cos t\sin t)\,dt}$$

$$= \frac{a\int_0^{\pi/2}\cos^2 t\sin t\,dt}{\int_0^{\pi/2}\cos t\sin t\,dt}$$

$$= \frac{a\int_{\cos(0)}^{\cos(\pi/2)} u^2(-du)}{\int_{\cos(0)}^{\cos(\pi/2)} u(-du)}$$

$$= \frac{a\int_1^0 u^2(-du)}{\int_1^0 u(-du)}$$

$$= \frac{a\int_0^1 u^2\,du}{\int_0^1 u\,du}$$

$$= \frac{a\left(\frac{1}{3}\right)}{\frac{1}{2}}$$

$$= \frac{2a}{3}$$

Calculate the y-coordinate of the center of mass.

$$\bar{y} = \frac{\displaystyle\int y\,dm}{\displaystyle\int dm} = \frac{\displaystyle\int_C y(\rho\,ds)}{\displaystyle\int_C \rho\,ds} = \frac{\displaystyle\int_0^{\pi/2} y(t)[kx(t)y(t)]\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt}{\displaystyle\int_0^{\pi/2} kx(t)y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt}$$

$$= \frac{k\displaystyle\int_0^{\pi/2} (a\cos t)(a\sin t)^2\sqrt{(-a\sin t)^2 + (a\cos t)^2}\,dt}{k\displaystyle\int_0^{\pi/2} (a\cos t)(a\sin t)\sqrt{(-a\sin t)^2 + (a\cos t)^2}\,dt}$$

$$= \frac{\displaystyle\int_0^{\pi/2} (a^3\cos t\sin^2 t)\sqrt{a^2\sin^2 t + a^2\cos^2 t}\,dt}{\displaystyle\int_0^{\pi/2} (a^2\cos t\sin t)\sqrt{a^2\sin^2 t + a^2\cos^2 t}\,dt}$$

$$= \frac{\displaystyle\int_0^{\pi/2} (a^4\cos t\sin^2 t)\,dt}{\displaystyle\int_0^{\pi/2} (a^3\cos t\sin t)\,dt}$$

$$= \frac{a\displaystyle\int_0^{\pi/2} \cos t\sin^2 t\,dt}{\displaystyle\int_0^{\pi/2} \cos t\sin t\,dt}$$

$$= \frac{a\displaystyle\int_{\sin(0)}^{\sin(\pi/2)} u^2\,du}{\displaystyle\int_{\sin(0)}^{\sin(\pi/2)} u\,du}$$

$$= \frac{a\displaystyle\int_0^1 u^2\,du}{\displaystyle\int_0^1 u\,du}$$

$$= \frac{a\left(\frac{1}{3}\right)}{\frac{1}{2}}$$

$$= \frac{2a}{3}$$

Therefore, the center of mass of the circular wire in the first quadrant is

$$\left(\frac{2a}{3}, \frac{2a}{3}\right).$$